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MOVEMENT OF THE SURFACE OF A COMPOUND HALF-SPACE IN A DYNAMIC SHEAR RUPTURE ALONG THE INTERNAL BOUNDARY

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A plane problem on a shear crack propagating in an unbounded medium was investigated in [1]. The movement of the surface of a half-space when a shear crack propagates at an angle to the surface was studied in [2]. Eexpressions for the accelerations and displacements of points on the free surface of a half-space were obtained for straight waves generated by the crack.

In this article, we consider dynamic phenomena in a half-space composed of two parts, rigid and elastic. Initially, adhesion between them is complete. A half-plane which is inclined at an arbitrary angle α to the free surface of the compound half-space is the interface. Movement of the half-space's surface is determined by an internal dynamic source, namely, a shear rupture which appears and is then located in the plane interface.

Such an idealization allows us to describe the processes connected with destruction of the layers falling down steeply, which takes place under a large difference in the rigidity characteristics of the materials in contact. The aim of this article is to evaluate the overloads on the surface of the half-space caused by waves falling down from the dynamic shear rupture.

We choose a point of the interface at a depth h as the origin of the Cartesian coordinates. The axes are oriented as shown in Fig. 1. Assume the z-axis to lie in the interface and to be parallel to the half-space's surface. Suppose that there is no deformation along the z-axis, so that the desired solution is a function of x, y, and t, i.e., a plane deformation is investigated. In the elastic part of the half-space, an initial static stress state is given, taking into account both the weight of the material and probable additional compression at infinity. The displacements in the contact plane are zero.

Let a rigid contact be broken at the moment $t = 0$ on the line $x = 0$, $y = 0$ $(-\infty < z < \infty)$ and shear stresses be large enough for a rupture to appear along the boundary. If X, Y are the Cartesian coordinates connected with the interface (see Fig. 1), then the current position of a defect is specified by the inequalities $-v_1 t < X < v_2 t$, $Y = 0$ $(-\infty < z < \infty)$. The constants v_1 and v_2 are the subsonic velocities of the edges of the rupture. Denote the known initial state in the X, Y-axes as

$$\sigma^0_{YY} = \sigma_0(X,Y) < 0, \qquad \sigma^0_{XY} = \tau_0(X,Y) > 0. \tag{1}$$

It is assumed that a dynamic stress relief proportional to the normal stress takes place on the shear rupture with closed faces (the lower is fixed, and the upper slides over it):

$$\sigma_{XY}(t,X,0) = -k\sigma_{YY}(t,X,0) \qquad (k = \text{const}, \quad k > 0),$$

where $\sigma_{ij}(t,X,0)$ are the stresses on the shift. The stress field $\sigma_{ij}(t,X,Y)$ changed by the shift can be represented as the sum of the initial static (1) and the dynamic stress $p_{ij}(t,X,Y)$:

$$\sigma_{ij} = \sigma^0_{ij} + p_{ij}.$$

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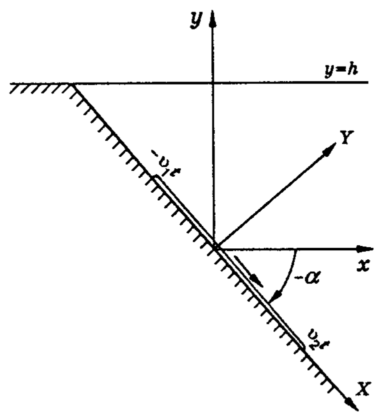

Fig. 1

The following conditions should be met for the desired dynamic state on the rupture and on the contact boundary with $t < h/(v_1 \sin\alpha)$ until the edge of the rupture reaches the half-space's surface:

$$
\begin{gathered}
p_{XY}(t,X,0) = -\tau_0(X,0) - k\sigma_0(X,0) - kp_{YY}(t,X,0) \quad (-v_1 t < X < v_2 t)\,, \\
U(t,X,0) = 0 \quad (X < -v_1 t,\ X > v_2 t), \qquad V(t,X,0) = 0 \quad (-h/\sin\alpha < X < \infty)\,.
\end{gathered} \tag{2}
$$

Here U and V are components of the displacement vector.

Note that under the conditions of a dry friction on the rupture two unknown stresses are not given in an explicit form but are only related to each other. If $k = 0$, conditions (2) describe the shear rupture without friction on the surface. In this case the shear stress totally dropped is equal to $-\tau_0$. Thus, conditions (2) determine the source of disturbances implicitly. The source will be known and can be prescribed by the displacements if we find the shift $U(t,X,0)$ on the shear rupture.

There is no load on the surface $y = h$, so the stresses in the x,y-axes are zero on it:

$$
\sigma_{yy}(t,x,h) = \sigma_{xy}(t,x,h) = 0. \tag{3}
$$

The boundary value problem (2), (3) is completed with zero initial conditions. The displacements are measured from those achieved in the equilibrium state, which has no effect on the boundary conditions (2).

We consider the boundary-value problem (2) for the moments of time until the compressional wave radiated by a defect and reflected from the half-space's surface reaches the shear rupture surface. We can find the source of the disturbances, the shift under the shear rupture, in this time interval, if we solve the problem of a dynamic shear rupture propagating along the boundary of contact between the elastic and rigid half-space, i.e., without considering the interaction of waves reflected from the surface of the elastic half-space and the defect. For the later moments of time, neglect of such waves in (2) is partly justified if the distance from the shear rupture to the surface is great, and therefore the influence of back waves on the defect and all the more on the movement of the half-space's surface under the subsequent reflections is insignificant due to the decrease in the amplitude of the wave as it travels.

On using the integral Laplace transform L over time and the Fourier transform F over the X-coordinate (s and q are the parameters of the transforms), we obtain the relation between the transforms of the displacements and the components of the vector of stresses on the boundary [3]:

$$
\begin{gathered}
\mu U^{LF}(s,q,0) = S_{11} p_{XY}^{LF}(s,q,0) + S_{12} p_{YY}^{LF}(s,q,0)\,, \\
\mu V^{LF}(s,q,0) = -S_{12} p_{XY}^{LF}(s,q,0) + S_{22} p_{YY}^{LF}(s,q,0)\,,
\end{gathered} \tag{4}
$$

$$S_{11} = -\frac{a_2^2 s^2 n_2}{R}, \quad S_{22} = -\frac{a_2^2 s^2 n_1}{R}, \quad S_{12} = -\frac{iq(n_2^2 + q^2 - 2n_1 n_2)}{R},$$

$$R = (n_2^2 + q^2)^2 - 4q^2 n_1 n_2, \quad n_i = \sqrt{q^2 + a_i^2 s^2} \qquad (i = 1, 2).$$

Here μ is the shear modulus, a_1 and a_2 are the reciprocals of the velocities of the compressional and shear wave in the deformable part of the half-space.

We search for shear stresses as the sum

$$p_{XY}(t, X, Y) = -k p_{YY}(t, X, Y) + f(t, X, Y). \tag{5}$$

With such a choice of p_{XY}, the boundary value of a new function f on the shear rupture is determined, and, as follows from (2), it has the form (in what follows, we omit the third argument of the functions at the boundary points)

$$f(t, X) = -\tau_0(X) - k\sigma_0(X) \qquad (-v_1 t < X < v_2 t). \tag{6}$$

After substitution of (5) and with regard for the condition $V = 0$ which is met over the contact boundary of the half-space, relationships (4) transform into the following:

$$\begin{gathered} \mu s U^{LF}(s, q) = \frac{s}{iq} W\Big(\frac{s}{iq}\Big) f^{LF}(s, q), \\ W\Big(\frac{s}{iq}\Big) = \frac{iq(q^2 - n_1 n_2)}{D}, \qquad D = ikq(n_2^2 + q^2 - 2n_1 n_2) + a_2^2 s^2 n_1. \end{gathered} \tag{7}$$

Only one boundary condition $V = 0$ is as yet satisfied in expressions (7) on the contact boundary. The conditions on the extension of the shear rupture for the displacement U and on the shear rupture for the function f should be taken into account in the first expression from (7) which connects the transforms of the functions mentioned, and thus we have the equation for finding the functions U and F.

In the time interval considered there will be no typical units of length and time in the problem if the vector of stresses of the initial static state is approximated by the sums

$$\sigma_0(X) = \sum_{n=0}^{N} \sigma_n^0 X^n, \quad \tau_0(X) = \sum_{n=0}^{N} \tau_n^0 X^n \qquad (\sigma_n^0 = \text{const}, \ \tau_n^0 = \text{const}) \tag{8}$$

in the contact line.

Suppose that this approximation takes place and the coefficients σ_n^0 and τ_n^0 are known. Then a solution of the dynamic problem for the contact line can be represented as the sum of the solutions to self-similar problems

$$p_{yy} = \sum_{n=0}^{N} p_{n\,yy}(X/t)\, t^n, \quad p_{xy} = \sum_{n=0}^{N} p_{n\,xy}(X/t)\, t^n, \quad f = \sum_{n=0}^{N} f_n(X/t)\, t^n,$$

whose homogeneity degree n is related to the number of the corresponding boundary value from (8), since, for example,

$$\sigma_0(X) = \sum_{n=0}^{N} \sigma_n^0 \left(\frac{X}{t}\right)^n t^n.$$

Thus, for every n the solution is reduced to the solution of (7) with conditions derived from (2), (8) (from here on the subscript n of functions is omitted for brevity):

$$f(t, X) = -(\tau_n^0 + k\sigma_n^0) X^n \quad (-v_1 t < X < v_2 t), \qquad U(t, X) = 0 \quad (X < -v_1 t, \ X > v_2 t). \tag{9}$$

The dependence of f on t, X is as follows: $f(t,X) = t^n\, Q(X/t)$.

We search for the solution of self-similar problems (7), (9) as in [3, 4]. The Laplace and Fourier transform of functions of the form $t^n\, Q(X/t)$ is a homogeneous function of the parameters of the transforms

$$f^{LF} = s^{-n-2} Q_0(s/iq)\,.$$

To invert homogeneous transforms we use the formula (see [3, 5])

$$f(t,X) = \hat{f}_+(t,X) - \hat{f}_-(t,X)\,, \quad \hat{f}(t,z) = -\frac{1}{2\pi i z n!}\int_0^t (t-\tau)^n Q_0(z/\tau)\,d\tau \quad (z = X + i\xi)\,. \tag{10}$$

Here the piecewise analytic function $\hat{f}(t,z)$ is an analytic representation of the function $f(t,X)$ [6], and the subscripts $\pm$ denote the limiting values of this function as $z \to X \pm i0$. As follows from (10), the jump in these values is equal to $f(t,X)$.

After the inversion of (7) by formula (10), we obtain an equation for the analytic representations of the functions U and f:

$$\mu \hat{U}^{(n+2)}(t,z) = \frac{z}{t}\, W(z/t)\hat{f}^{(n+1)}(t,z)\,, W(z/t) = \frac{1 + n_1 n_2}{k\,[2 + 2n_1 n_2 - (a_2 z/t)^2] - (a_2 z/t)^2 n_1}\,, \tag{11}$$

$$n_i = \sqrt{(a_i z/t)^2 - 1}\,, \quad n_i > 0 \qquad (X/t > a_i^{-1}\,,\ \xi = 0,\ i = 1,2)$$

(the superscripts in parentheses denote the order of a partial derivative with respect to time). Single-valued branches of the radicals n_i on the plane z/t are marked with cuts $(-a_i^{-1}, a_i^{-1})$ along the real axis, and in this case the values of radicals on the real axis are determined as

$$n_i = \begin{cases} \operatorname{sgn}(X)\sqrt{(a_i X/t)^2 - 1} & (|X/t| > a_i^{-1},\ \xi = 0)\,, \\ \pm i\sqrt{1 - (a_i X/t)^2} & (|X/t| < a_i^{-1},\ \xi = \pm 0)\,. \end{cases}$$

The solution of Eq. (7) with respect to the transforms is reduced to the solution of Eq. (11) with respect to two piecewise-analytic functions, and the jumps in their time derivatives are determined from (9):

$$\begin{aligned} U^{(n+2)} &= \hat{U}_+^{(n+2)} - \hat{U}_-^{(n+2)} = 0 \qquad (X < -v_1 t,\ X > v_2 t)\,, \\ f^{(n+1)} &= \hat{f}_+^{(n+1)} - \hat{f}_-^{(n+1)} = 0 \qquad (-v_1 t < X < v_2 t)\,. \end{aligned} \tag{12}$$

The solution of Eq. (11) with conditions (12) is found with the help of the auxiliary function

$$w(z/t) = (z/t + v_1)\exp\left[-\frac{1}{\pi}\int_{-v_1}^{v_2} \arctan \varphi(\eta)\,\frac{d\eta}{\eta - z/t}\right],$$

$$\varphi(\eta) = \frac{a_2^2\eta^2\sqrt{1 - a_1^2\eta^2}}{k\left[2 - 2\sqrt{1 - a_1^2\eta^2}\sqrt{1 - a_2^2\eta^2} - a_2^2\eta^2\right]}\,, \tag{13}$$

which is constructed so that the jump in the function

$$\Omega(t,z) = \frac{t}{z}\, w\left(\frac{z}{t}\right)\hat{U}^{(n+2)} = w\left(\frac{z}{t}\right) W\left(\frac{z}{t}\right)\hat{f}^{(n+1)}$$

can be found at each point of the real axis [7].

As follows from (9), the jump in Ω is zero throughout the axis, except for the points of change of the boundary conditions, and hence the function has the form ([3])

$$\Omega(t,z) = \frac{1}{2\pi i}\sum_{j=0}^{n}\left[\frac{t^j A_j}{(z + v_1 t)^{j+1}} + \frac{t^j B_j}{(z - v_2 t)^{j+1}}\right],$$

i.e., the jump in Ω is a function located at the points $X = v_2 t$ and $X = -v_1 t$.

Taking into account (13), we obtain a solution to functional equation (11):

$$\mu\widehat{U}^{(n+2)}(t,z) = \frac{z}{t}\frac{\Omega(t,z)}{w(z/t)}, \qquad \widehat{f}^{(n+1)}(t,z) = \frac{\Omega(t,z)}{w(z/t)W(z/t)}.$$

The constants A_j and B_j are found from the boundary condition (9) with respect to f. Integrating $\widehat{f}$ over time we have

$$\begin{gathered}
f(t,X) = T_0\,H(v_1 t + X) + \frac{1}{n!}\int\limits_0^{-X/v_1}(t-\tau)^n\Phi_1\left(\frac{X}{\tau}\right)\frac{d\tau}{\tau} \qquad (-v_1 t < X < 0), \\
f(t,X) = T_0\,H(v_2 t - X) + \frac{1}{n!}\int\limits_0^{X/v_2}(t-\tau)^n\Phi_2\left(\frac{X}{\tau}\right)\frac{d\tau}{\tau} \qquad (0 < X < v_2 t), \\
T_0 = -(\tau_n^0 + k\sigma_n^0)X^n,
\end{gathered} \tag{14}$$

where $H(\ldots)$ are the Heaviside functions; Φ_1 and Φ_2 are the jumps in the function $\Phi(z/t) = t\Omega(t,z)/[w(z/t)W(z/t)]$ on the intervals $(-v_1 t < X < 0)$ and $(0 < X < v_2 t)$.

It is evident that to satisfy the boundary condition (9) in relation to f it is necessary to set the integrals in (14) equal to zero, and after a change of the integration variable they take the form

$$\int\limits_0^{1/v_1}\Phi_1\left(-\frac{1}{\eta}\right)\eta^{-m-1}d\eta = 0, \qquad \int\limits_0^{1/v_2}\Phi_2\left(\frac{1}{\eta}\right)\eta^{-m-1}d\eta = 0 \qquad (m = 0,1,\ldots,n).$$

We can find all the constants A_j and B_j from these equalities.

For simplicity we restrict ourselves only to the first terms of the stress expansion, i.e., from here on assume that $n = 0$. The constants A and B were calculated in [7] for this case:

$$\begin{gathered}
\Omega(t,z) = \frac{1}{2\pi i}\left(\frac{A}{z+v_1 t} + \frac{B}{z - v_2 t}\right), \qquad Av_2 - Bv_1 = 0, \qquad A + B = \pi(\tau_0 + k\sigma_0)/J, \\
J = \operatorname{Re}\int\limits_0^\infty \frac{d\xi}{(v_1+\gamma+i\xi)(v_2-\gamma-i\xi)w(\gamma+i\xi)W(\gamma+i\xi)}, \qquad \gamma = \frac{v_2 - v_1}{2}.
\end{gathered}$$

Thus, we have found the source of the disturbances. On solving the problem on the half-space $Y > 0$ under the conditions

$$U(t,X,0) = tu_0(X/t), \quad V(t,Y,0) = 0,$$

given on the surface $Y = 0$, we determine the displacements in the waves propagating toward the boundary $y = h$. To perform this we should find the LF-transform of the function U on the boundary. Note that when $n = 0$, the function U is a homogeneous function of degree -3

$$U^{LF} = s^{-3}U_0(s/iq),$$

and on the other hand, as follows from (10), it has the corresponding analytic representation

$$\begin{gathered}
\widehat{U}(t,z) = -\frac{1}{2\pi z}\int\limits_0^t (t-\tau)\,U_0(z/\tau)\,d\tau, \\
U_0(z/\tau) = -\frac{\pi C z/\tau}{(1+v_1\tau/z)(1-v_2\tau/z)w(z/\tau)}, \qquad C = \frac{A+B}{\pi\mu} = \frac{\tau_0 + k\sigma_0}{\mu J}.
\end{gathered}$$

By comparing the expressions obtained, we get

$$U_0(s/q) = \pi i C \frac{s/q}{(1 + iv_1 q/s)(1 - iv_2 q/s)\, w(s/iq)}.$$

Then we easily obtain the transforms of the displacements in a wave propagating through the half-space $Y > 0$:

$$s^3 U^{LF}(s, q, Y) = \frac{U_0(s/iq)}{q^2 - n_1 n_2} (q^2 \mathrm{e}^{-n_1 Y} - n_1 n_2 \mathrm{e}^{-n_2 Y}),$$

$$s^3 V^{LF}(s, q, Y) = -\frac{iqn_1 U_0(s/iq)}{q^2 - n_1 n_2} (\mathrm{e}^{-n_1 Y} - \mathrm{e}^{-n_2 Y}), \tag{15}$$

$$n_i = \sqrt{a_i^2 s^2 + q^2}.$$

It remains to take into account the interaction of the resultant waves with the boundary $y = h$. For this we solve the problem of reflection of waves from the free boundary of the half-space. To do this we need, first, to change from the X, Y coordinate system to the x, y system, then to recount displacements (15) by the formula $u + iv = (U + iV)\mathrm{e}^{i\alpha}$ and to perform the Fourier transform $\mathcal{F}$ over x (transform parameter p), relating it to the Fourier transform F over X in expressions (15), as in [2]. After this recalculation, on satisfying conditions (3), we have the $L\mathcal{F}$-transforms of the displacements in a wave, taking into account the reflection of a compressional and a shear wave from the boundary. On the boundary $y = h$, the transforms of the displacements have the form

$$s^3 u^{L\mathcal{F}}(s, p, h) = 4\, ipm_2 F_1(p/s)\mathrm{e}^{-m_1 h} + 2\,(a^2 s^2 + 2\,p^2) F_2(p/s)\mathrm{e}^{-m_2 h},$$

$$s^3 v^{L\mathcal{F}}(s, q, h) = 2\,(a_2^2 s^2 + 2\,p^2) F_1(p/s)\mathrm{e}^{-m_1 h} - 4\, ipm_1 F_2(p/s)\mathrm{e}^{-m_2 h},$$

$$F_1(p/s) = \frac{a_2^2 s^4 q_1\,(p \sin\alpha + im_1 \cos\alpha)}{[n_1(q_1) n_2(q_1) - q_1^2] R}\, U_0(s/iq_1),$$

$$F_2(p/s) = -\frac{ia_2^2 s^4 n_1(q_2)(p \sin\alpha + im_2 \cos\alpha)}{[n_1(q_2) n_2(q_2) - q_2^2] R}\, U_0(s/iq_2),$$

$$R = (m_2^2 + p^2)^2 - 4p^2 m_1 m_2, \quad m_j = \sqrt{a_j^2 s^2 + p^2}, \quad n_j(q_l) = \sqrt{a_j^2 s^2 + q_l^2}, \quad iq_l = p \cos\alpha - im_l \sin\alpha.$$

We invert these transforms which have homogeneous function-factors of the exponents with the help of analytic representations [3, 5]. After that, the vertical component of the acceleration on the surface results from the formulas

$$\ddot{v}(t, x, h) = \frac{2}{\pi} \mathrm{Re} \Big\{ (a_2^2 + 2\xi_1^2)\, F_1(\xi_1) \frac{\partial \xi_1}{\partial t} H(t - a_1 h) - 4i\xi_2 m_1(\xi_2) F_2(\xi_2) \frac{\partial \xi_2}{\partial t} H(t - a_2 h) \Big\},$$

$$m_j(\xi_l) = \sqrt{a_j^2 + \xi_l^2}.$$

Here ξ_j are the roots of the equation $h\sqrt{a_j^2 + \xi^2} - iz\xi = t \quad (z = x + i0)$:

$$\xi_j = -\frac{itx}{h^2 + x^2} + \frac{h}{h^2 + x^2} \begin{cases} \sqrt{t^2 - a_j^2(h^2 + x^2)}, & t^2 > a_j^2(h^2 + x^2), \\ i \,\mathrm{sgn}\,(x)\sqrt{a_j^2(h^2 + x^2) - t^2}, & t^2 < a_j^2(h^2 + x^2). \end{cases}$$

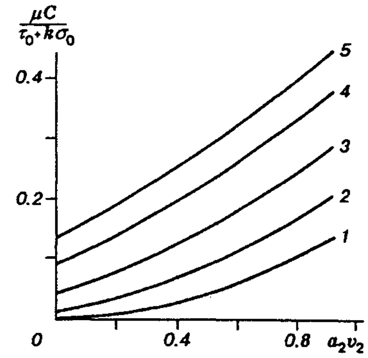

Fig. 2

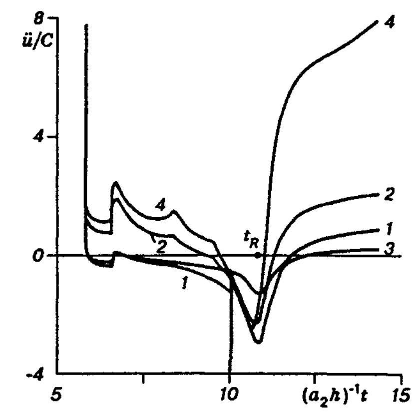

Fig. 3

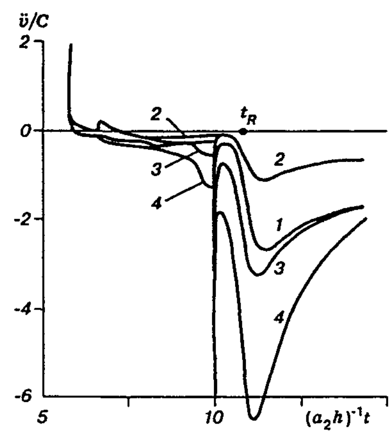

Fig. 4

The signs of the real and imaginary part of each calculated complex radical satisfy the inequalities

$$\operatorname{Re} m_j(\xi_i) \geqslant 0, \quad x \operatorname{Im} m_j(\xi_i) \leqslant 0.$$

Figures 2–4 present the calculations that were carried out for $k = 0$. The depth h at which the shear starts and the velocity of the shear wave a_2^{-1} were considered the units of measurement.

Figure 2 shows the dependence of C on v_2 with $a_2 v_1 = 0$, 0.25, 0.5, 0.75, 0.9 (curves *1–5*). Figures 3 and 4 present the horizontal $\ddot{u}$ and vertical $\ddot{v}$ components of the acceleration of a particle on the half-space's surface at the point with coordinates (10, 1) when $\alpha = 0$ (curve *1* is for $v_1 = 0.25$ and $v_2 = 0$, curve *2* is for $v_1 = 0$, $v_2 = 0.25$) and when $\alpha = -\pi/6$ (curve *3* is for $v_1 = 0.25$, $v_2 = 0$, curve *4* is for $v_1 = 0$, $v_2 = 0.25$). The point t_R corresponds to the moment when the Rayleigh wave arrives at the point of observation from the epicenter. Note that the direction of breakage can change the acceleration amplitudes by 1.5 to 2 times in the front of the Rayleigh wave.

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